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1998 J. Phys. A: Math. Gen. 31 3573

(http://iopscience.iop.org/0305-4470/31/15/020)

J. Phys. A: Math. Gen. **31** (1998) 3573–3593. Printed in the UK PII: S0305-4470(98)86391-9

Clebsch–Gordan problem for three-dimensional Lorentz group in the elliptic basis: I. Tensor product of continuous series

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Received 29 July 1997

Abstract. This paper is the first of two papers devoted to the study of the Clebsch–Gordan (CG) problem for the three-dimensional Lorentz group $SO_0(2,1)$ in an elliptic (or SO(2)) basis. Here we describe the reduction of the tensor product of two unitary irreducible representations (UIRs) of the continuous series, i.e. belonging to either the principal or complementary series. The corresponding CG coefficients are defined as matrix elements of an intertwining operator between the tensor product representation and the irreducible component appearing in the decomposition. We then obtain an expression for CG coefficients in terms of a single function, namely in terms of the bilateral series ${}_3H_3(1)$ with unit argument defined in the complex space C_3 of the variable j_1, j_2, j. In the general case the ${}_3H_3(1)$ functions are expressed in terms of two hypergeometric functions ${}_3F_2$ with unit argument; however, it reduces to the single ${}_3F_2(1)$ function if at least one of the coupling UIRs belong to a discrete series. We derive a completeness relation for CG coefficients for all the cases under consideration.

1. Introduction

This paper is the first of two devoted to the study of the Clebsch–Gordan (CG) problem for unitary irreducible representations (UIRs) of the three-dimensional Lorentz group $SO_0(2,1)$ in an SO(2) basis. In the present paper we describe the reduction of the tensor product of two UIRs of the continuous series, i.e. belonging to either the principal or complementary series. The tensor product of the remaining cases will be studied in the next paper of this series.

The three-dimensional Lorentz group $SO_0(2,1)$ is the most important non-compact Lie group used so far in mathematics and physics. The UIRs of $SO_0(2,1)$ or its double covering group SU(1,1) were given by Bargmann [1] many years ago. Since the advent of $SO_0(2,1)$ symmetry in relativistic scattering theory and in dynamical symmetry group theory, it has become necessary to know the CG coefficients for this group.

The CG coefficients of $SO_0(2,1)$ in an SO(2) basis have already been dealt with in certain cases; if the two representations both belong to the positive discrete series, or if both belong to the negative discrete series, then the CG coefficients have been worked out by Andrews and Gunson [2] and Sannikov [3]. Holman and Biedenharn [4] derived many CG coefficients solving a second-order finite-difference equation. Thus their CG coefficient is not analytically containable to other cases of coupling. Ferreti and Verde [5] worked out

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the CG series for the tensor product of two principal series with some restrictions on the 'magnetic' quantum numbers by using an indirect method initiated by Andrew and Ganson. This method was later used by Wang [6] who restated their results and also worked on the remaining cases except that of the complementary series. An (incomplete) study of the CG problems for the $SO_0(2,1)$ group in an SO(2) basis is also presented in [7–9].

No author, to the best of our knowledge, has explicitly calculated the CG coefficients for all possible cases of tensor products of UIRs of $SO_0(2,1)$ in an SO(2) basis. This paper fills these gaps. The purpose of the present work is to determine the CG coefficients, in a direct and rigorous manner. We suggest a new method, in which the CG coefficients are defined as matrix elements of an intertwining operator between the tensor product representation and the irreducible component appearing in the decomposition. Our task is greatly simplified by the fact that the intertwining operators and the Plancheral formulae for all tensor products of UIRs of $SO_0(2,1)$ have already been obtained by Molchanov [10] (for a review, see also [11, 12]).

The contents of this paper are arranged as follows. In section 2 we present all the mathematical preliminaries and notation necessary for subsequent sections. In section 3 the decomposition of the tensor product of the two principal series representations onto irreducible representations is described. The corresponding CG coefficients are defined as matrix elements of operators intertwining the tensor product of principal series and irreducible components appearing in the decomposition. We then obtain an expression for CG coefficients in terms of a single function, namely in terms of the bilateral series ${}_3H_3(1)$ with unit argument defined in the complex space C_3 of the variable j_1, j_2, j. This function is singular on a subset of discrete points of C_3, corresponding to the case when all three UIRs belong to the discrete series. In the general case the ${}_3H_3(1)$ function is expressed in terms of two hypergeometric functions ${}_3F_2$ with unit argument; however, it reduces to the single ${}_3F_2(1)$ function if at least one of the coupling UIRs belong to a discrete series.

In section 4 we discuss the tensor product of the complementary series representation with a representation of the principal series. The tensor product of pairs of complementary series representations is studied in section 5. We derive completeness relations for CG coefficients for all these cases under consideration. Some mathematical results necessary for section 3 are given in appendices A–C.

2. The group $SO_0(2,1)$

In this section we establish notation and review those properties of $G = SO_0(2,1)$ that we will need later. For a more detailed treatment of G we refer to [1, 11–13].

The group G is the connected component of the group of proper linear transformations of a three-dimensional pseudo-Euclidean space $R^{2,1}$ which preserves the bilinear form

$$[x, y] = x_1y_1 + x_2y_2 - x_3y_3. \tag{2.1}$$

Every element g of G can uniquely be factorized into

$$g = h_\tau a_\eta k_\theta \tag{2.2}$$

each factor constituting a sub-group of G. They are explicitly given by

$$h_\tau = \begin{pmatrix} \cosh\tau & 0 & \sinh\tau \\ 0 & 1 & 0 \\ \sinh\tau & 0 & \cosh\tau \end{pmatrix} \in \mathsf{H} \qquad a_\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\eta & \sinh\eta \\ 0 & \sinh\eta & \cosh\eta \end{pmatrix} \in \mathsf{A}$$

and

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{K}.$$

We shall consider G as acting on $R^{2,1}$ on the right. In accordance with this we shall write the vector in row form.

The group G acts transitively on the one-sheeted hyperboloid $X = \{x \in R^{2,1} : [x, x] = 1\}$. An invariant measure on X is $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2/|x_3|$. The decomposition (2.2) tells us that the hyperboloid X can be parametrized by coordinates η, θ as follows

$$x = \overset{0}{x}\,\alpha_x \qquad \overset{0}{x} = (0, 1, 0)$$

where $\alpha_x = a_\eta k_\theta$. It is also worth noting that X is isomorphic to the co-set space $\mathsf{H}\backslash G$.

The generators of the Lie algebra of G are denoted by J_1, J_2, J_3. Here J_1, J_2 are the generators of the pure Lorentz transformations h_τ, a_η along the 1 and 2 axes, respectively, and J_3 is the generator of rotations k_θ in the 1,2 plane. The unitary faithful irreducible representations of G are infinite-dimensional. All such UIRs are labelled by the eigenvalue of the Casimir operator $\mathsf{Q} = \mathsf{J}_1^2 + \mathsf{J}_2^2 - \mathsf{J}_3^2 = -j(j+1)$, where J_1, J_2, J_3 are the Hermitian operators corresponding to J_1, J_2, J_3, respectively, in the Lie algebra representation. J_3 is elliptic, J_1, J_2 hyperbolic. When we use a SO(2) basis, J_3 will be the preferred generator. The eigenvalues of J_3 will be denoted by m.

We now give the spectrum of j corresponding to UIRs and eigenvalues m of the operator J_3 in each such representation. (For the purposes of this paper we only consider the single-valued representations.)

(i) Principal series $T_{\mathrm{i}\rho-\frac{1}{2}}$:

$$j = -\tfrac{1}{2} + \mathrm{i}\rho \qquad 0 \leqslant \rho < \infty \qquad m = 0, \pm 1, \pm 2, \ldots.$$

(ii) Complementary series T_τ:

$$j = \tau \qquad -1 < \tau < -\tfrac{1}{2} \qquad m = 0, \pm 1, \pm 2, \ldots.$$

(iii) Positive discrete series T_l^+:

$$j = l \qquad l = 0, 1, 2, \ldots \qquad m = l+1, l+2, \ldots.$$

(iv) Negative discrete series T_l^-:

$$j = l, l = 0, 1, 2, \ldots \qquad m = -l-1, -l-2, \ldots.$$

Any UIR of G is equivalent to some sub-representation of an elementary representation T_j, $j \in C$. They occur as unitarizations of elementary representations or as unitarizations of quotients of such representations.

Let us recall some facts about the elementary representations of the group G. The representations T_j, $j \in C$, can be realized in the space of the infinitely differentiable function $f(x)$ on the upper sheet of the two-dimensional cone $x_1^2 + x_2^2 - x_3^2 = 0$, $x_3 > 0$, homogeneous of degree j

$$f(ax) = a^j f(x) \qquad a > 0. \tag{2.3}$$

The representations T_j are given by

$$T_j(g) f(x) = f(xg)$$

where $g \in G$.

Generally we may choose a large number of different coordinate systems on the cone. The different choices of coordinate systems on the cone lead to different reductions of the group G to its sub-group. The SO(2) basis is given by the decomposition according to the compact sub-group $SO_0(2,1) \supset SO(2)$. As a prelude to this decomposition one introduces the spherical coordinates on the cone given by $x = r(\cos\varphi, \sin\varphi, 1)$, where $0 \leqslant r < \infty$, $0 \leqslant \varphi < 2\pi$. From (2.3) it follows that the homogeneous function is defined uniquely by its values on the circle $S^1 = \{s = (\cos\varphi, \sin\varphi, 1)\}$ in $R^{2,1}$. Consequently, elementary representations of G can be realized on the space D_j of infinitely differentiable functions $f(s)$ on S^1. In this realization the representations of G are given by

$$(T_j(g)f)(s) = (sg)_3^j f(s_g) \qquad g \in G \tag{2.4}$$

where $s_g = (sg)/(sg)_3$. (For the sake of simplicity, the realization of T_j in the space of the function in S^1 is denoted by the same symbol T_j and the restriction of a function f onto S^1 is denoted by the same symbol f.) The operator B_j in D_j defined by

$$B_j f(s) = \frac{2^j \Gamma(j+1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}-j)} \int_{S^1} |[s,t]|^{-j-1} f(t)\,\mathrm{d}t \tag{2.5}$$

intertwines T_j and T_{-1-j}: $T_{-1-j}(g)B_j = B_j T_j(g)$. If j is not an integer, then T_j is irreducible and is equivalent to T_{-1-j}. When $j = l$, $l = 0, 1, 2, \ldots$ in D_l there are three invariant sub-spaces:

(i) D_l^+, the sub-space of D_l consisting of functions of the form

$$f(s) = \sum_{n=-l}^{\infty} a_n \,\mathrm{e}^{\mathrm{i}n\varphi} \qquad s = (\cos\varphi, \sin\varphi, 1)$$

(ii) D_l^-, the sub-space of functions of the form

$$f(s) = \sum_{n=-\infty}^{l} a_n \,\mathrm{e}^{\mathrm{i}n\varphi} \qquad s = (\cos\varphi, \sin\varphi, 1)$$

(iii) $D_l^0 = D_l^+ \cap D_l^-$, the sub-space spanned by $\mathrm{e}^{\mathrm{i}m\varphi}$, $-l \leqslant m \leqslant l$.

The representations of G, induced by T_l in the sub-space D_l^0 and in the factor spaces D_l^+/D_l^0 and D_l^-/D_l^0 are irreducible.

The representations just described give three series of unitarizable representations.

(i) $j = -1/2 + \mathrm{i}\rho$, $0 \leqslant \rho < \infty$. In this case

$$(f_1, f_2)_{\mathrm{i}\rho-1/2} = \int_{S^1} f_1(s)\overline{f_2(s)}\,\mathrm{d}s \tag{2.6}$$

defines a scalar product in D_j. Here $\mathrm{d}s$ is the Euclidean measure on S^1 and the bar means complex conjugation. This makes (2.4) a unitary representation which is also irreducible. This unitary representation forms the principal series $T_{\mathrm{i}\rho-1/2}$ of UIRs of G. The completion of D_j with respect to the norm induced by the scalar product (2.6) yields the Hilbert space $H_{\mathrm{i}\rho-1/2} = L^2(S^1)$ of a square integrable function over S^1. We shall identify the representations T_j, $j = -1/2 + \mathrm{i}\rho$, with its extension to a UIR of G in $H_{\mathrm{i}\rho-1/2}$. The vectors of the SO(2) basis $f_m^{\mathrm{i}\rho-1/2}$ in $H_{\mathrm{i}\rho-1/2}$ are

$$f_m^{\mathrm{i}\rho-1/2}(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{\Gamma(\frac{1}{2}-\mathrm{i}\rho+m)}{\Gamma(\frac{1}{2}+\mathrm{i}\rho+m)}\right]^{1/2} \mathrm{e}^{\mathrm{i}m\varphi} \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{2.7}$$

(ii) $j = \tau, -1 < \tau < -\frac{1}{2}$. The scalar product in this case is given by

$$(f_1, f_2)_\tau = \frac{2^\tau}{\sqrt{\pi}} \frac{\Gamma(\tau+1)}{\Gamma(-(2\tau+1)/2)} \int_{S^1\times S^1} |[s,t]|^{-\tau-1} f_1(s)\overline{f_2(t)}\,\mathrm{d}s\,\mathrm{d}t. \tag{2.8}$$

The Hilbert space completion of D_τ with respect to the scalar product (2.8) will be denoted by H_τ. We shall identify the representation T_j (for $j = \tau$) with its extension to a UIR of G in H_τ. The family of UIRs so constructed form the complementary series. The functions

$$f_m^\tau(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{\Gamma(-\tau+m)}{\Gamma(1+\tau+m)}\right]^{1/2} \mathrm{e}^{\mathrm{i}m\varphi} \qquad -1 < j < 0, m = 0, \pm 1, \pm 2, \ldots \tag{2.9}$$

form an orthonormal basis in H_τ.

(iii) $j = l$, $l = 0, 1, 2, \ldots$. In this case the representations T_l^+ and T_l^- induced by T_l in the quotient spaces D_l^+/D_l^0 and D_l^-/D_l^0 are unitary with respect to the scalar product

$$(\tilde{f}_1, \tilde{f}_2)_{l+} = \sum_{n=l+1}^{\infty} \frac{\Gamma(1+l+n)}{\Gamma(-l+n)} a_n \overline{b_n} \tag{2.10}$$

and

$$(\tilde{f}_1, \tilde{f}_2)_{l-} = \sum_{n=-\infty}^{-l-1} \frac{\Gamma(1+l-n)}{\Gamma(-l-n)} a_n \overline{b_n} \tag{2.11}$$

respectively, where $\tilde{f}_i = f_i + D_l^0$ and a_n, b_n are defined by

$$f_1(s) = \sum_{n=-\infty}^{\infty} a_n\, \mathrm{e}^{\mathrm{i}n\varphi} \qquad f_2(s) = \sum_{n=-\infty}^{\infty} b_n\, \mathrm{e}^{\mathrm{i}n\varphi}.$$

These representations form the discrete series of UIRs of G. The Hilbert space completion of D_l^+/D_l^0 and D_l^-/D_l^0 with respect to (2.10) and (2.11) will be denoted by H_l^+ and H_l^-, respectively.

The functions $\tilde{f}_m^{l\pm}$, where

$$f_m^{l\pm}(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{\Gamma(-l\pm m)}{\Gamma(1+l\pm m)}\right]^{1/2} \mathrm{e}^{\mathrm{i}m\varphi}$$
$$m = l+1, l+2, \ldots, (m = -l-1, -l-2, \ldots) \tag{2.12}$$

form an orthonormal basis in $H_l^+(H_l^-)$.

3. The tensor product of two principal series representations

As was pointed out in the introduction, the CG coefficients can be defined as matrix elements of an intertwining operator between the tensor product representation and the irreducible component appearing in the decomposition. Therefore, we start by quoting the results of Molchanov (for details see [10] and references therein).

The tensor product $T_{j_1} \otimes T_{j_2}$, $j_k = -1/2 + \mathrm{i}\rho_k$, $k = 1, 2$, of two principal series representations of G can be realized on the Hilbert space $L^2(S^1 \times S^1) = L^2(S^1) \otimes L^2(S^1)$ of the square integrable functions f over $S^1 \times S^1$ with the scalar product

$$(f_1, f_2)_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2} = \int_{S^1\times S^1} f_1(s,t)\overline{f_2(s,t)}\, \mathrm{d}s\, \mathrm{d}t \tag{3.1}$$

where $s = (\cos\varphi_1, \sin\varphi_1, 1)$, $t = (\cos\varphi_2, \sin\varphi_2, 1)$. A convenient basis of $L^2(S^1 \times S^1)$ is given by

$$f_{m_1m_2}^{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(s,t) = \frac{1}{2\pi}\left[\frac{\Gamma(\frac{1}{2}-\mathrm{i}\rho_1+m_1)\Gamma(\frac{1}{2}-\mathrm{i}\rho_2+m_2)}{\Gamma(\frac{1}{2}+\mathrm{i}\rho_1+m_1)\Gamma(\frac{1}{2}+\mathrm{i}\rho_2+m_2)}\right]^{1/2} \mathrm{e}^{\mathrm{i}m_1\varphi_1+\mathrm{i}m_2\varphi_2}. \tag{3.2}$$

The representation $T_{j_1} \otimes T_{j_2}$ acts on the Hilbert space $L^2(S^1 \times S^2)$ by

$$(T_{j_1} \otimes T_{j_2}(g) f)(s,t) = (sg)_3^{j_1} (tg)_3^{j_2} f(s_g, t_g) \qquad g \in G \tag{3.3}$$

where $s_g = (sg)/(sg)_3$ and $t_g = (tg)/(tg)_3$.

Let $\sigma \in C$ and let $\chi_\sigma(h_\tau) = \mathrm{e}^{\sigma\tau}$ be a one-dimensional representation of H. Let U_σ be a representation of G induced by χ_σ. Then U_σ can be realized on the Schwartz space $K(X)$ of functions $\Phi(x)$ on $X = \mathsf{H}\backslash G$, where G acts as follows

$$(U_\sigma(g)\Phi)(x) = \chi_\sigma(\alpha_x g \alpha_{xg}^{-1})\Phi(xg). \tag{3.4}$$

When $\sigma \in \mathrm{i}R$, the representation (3.4) can be extended to a unitary representation of G in $L^2(X)$.

The representation $T_{j_1} \otimes T_{j_2}$, $j_1, j_2 \in C$, is Naimark (or infinitesimally) equivalent to the representation $U_{j_2 - j_1}$. The equivalence is effected by the map $P_{j_1 j_2}$: $K(S^0) \to K(X)$ defined as

$$\Phi(x) = (P_{j_1 j_2} f)(x) = (-[s,t])^{-(j_1+j_2)/2} f(s,t) \tag{3.5}$$

for

$$x = -\frac{1}{[s,t]}(s_2 - t_2, s_1 - t_1, s_2 t_1 - s_1 t_2)$$

where S^0 is the complement of diagonal S_0 in $S^1 \times S^1$. When $j_k = -1/2 + \mathrm{i}\rho_k$ $(k = 1, 2)$, $P_{j_1 j_2}$ can be extended to a unitary map $L^2(S \times S^1) \to L^2(X)$, such that

$$P_{j_1 j_2}(T_{j_1} \otimes T_{j_2}) = U_{j_1 - j_2} P_{j_1 j_2}. \tag{3.6}$$

So the problem of decomposing the tensor product of two principal series representations is reduced to that of decomposing $U_{j_2 - j_1}$.

Let $Q_{j\varepsilon}^z$ be an operator from $K(X)$ into D_j defined by

$$(Q_{j\varepsilon}^z \Phi)(s) = \gamma(z, j, \varepsilon) \int_X \Phi(x) |[x,s]|^{j-2\varepsilon} [x,s]^{2\varepsilon} ([s^- \alpha_x, s]/[s^+ \alpha_x, s])^{z/2}\, \mathrm{d}x \tag{3.7}$$

where $\Phi \in K(X)$, $s^\pm = (\pm 1, 0, 1)$, $\varepsilon \in \{0, 1/2\}$, $\mathrm{Re}\, j > |\mathrm{Re}\, z| - 1$ and $\gamma(z, j, \varepsilon)$ is given by

$$\gamma(z, j, \varepsilon) = \left[\Gamma\left(\frac{z+j+1}{2} + \varepsilon\right)\Gamma\left(\frac{-z+j+1}{2} + \varepsilon\right)\right]^{-1}. \tag{3.8}$$

It can be shown that the integral in (3.7) may be continued analytically in j and z to give the entire function. Moreover,

$$Q_{j\varepsilon}^z U_z(g) = T_j(g) Q_{j\varepsilon}^z \tag{3.9}$$

where $T_j(g)$ is the elementary representation of G. Therefore, an intertwining operator $C_{j_1 j_2}^{j\varepsilon}$ between $T_{j_1} \otimes T_{j_2} (j_1, j_2 \in C)$ and T_j is defined as a composition of maps $Q_{j\varepsilon}^{j_2 - j_1}$ and $P_{j_1 j_2}$

$$(C_{j_1 j_2}^{j\varepsilon} f)(u) = \gamma(j_2 - j_1, j, \varepsilon) \int_{S^1 \times S^1} K_\varepsilon(j_1 s, j_2 t; ju) f(s,t)\, \mathrm{d}s\, \mathrm{d}t. \tag{3.10}$$

Here

$$K_\varepsilon(j_1 s, j_2 t; ju) = 2^{-1-((j_1+j_2)/2)} \left|\sin\frac{\varphi_1 - \varphi}{2}\right|^{-2a_1} \mathrm{sign}^{2\varepsilon} \sin\frac{\varphi_1 - \varphi}{2} \left|\sin\frac{\varphi - \varphi_2}{2}\right|^{-2a_2}$$
$$\times \mathrm{sign}^{2\varepsilon} \sin\frac{\varphi - \varphi_2}{2} \left|\sin\frac{\varphi_1 - \varphi_2}{2}\right|^{-2a} \mathrm{sign}^{2\varepsilon} \sin\frac{\varphi_1 - \varphi_2}{2} \tag{3.11}$$

where $s = (\cos\varphi_1, \sin\varphi_1, 1)$, $t = (\cos\varphi_2, \sin\varphi_2, 1)$, $u = (\cos\varphi, \sin\varphi, 1)$ and

$$-2a = -2 - j_1 - j_2 - j \qquad -2a_1 = z + j \qquad -2a_2 = -z + j \qquad z = j_2 - j_1. \tag{3.12}$$

The map $C^{j\varepsilon}_{j_1 j_2}$ induces a unitary operator from $L^2(S^1\times S^1)$ into a direct integral of Hilbert spaces supporting the unitary representations $T_{\mathrm{i}\rho-1/2}$ and $T_l^+ \oplus T_l^-$ (see later). Therefore, in the case when the two representations both belong to the principal series, the structure of the CG series is given by

$$T_{\mathrm{i}\rho_1-1/2} \otimes T_{\mathrm{i}\rho_2-1/2} = 2\int_0^\infty T_{-(1/2)+\mathrm{i}\rho}\,\mathrm{d}\rho \oplus \sum_{l=0}^{\infty}(T_l^+ \oplus T_l^-). \tag{3.13}$$

In other words, the tensor product contains two copies of the principal series and one of each discrete series representations.

The space of operator $C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}$ intertwining $T_{\mathrm{i}\rho_1-1/2} \otimes T_{\mathrm{i}\rho_2-1/2}$ and $T_{\mathrm{i}\rho-1/2}$

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(T_{\mathrm{i}\rho_1-1/2} \otimes T_{\mathrm{i}\rho_2-1/2}(g)) = T_{\mathrm{i}\rho-1/2}(g)C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2} \tag{3.14}$$

is two-dimensional, where $\varepsilon = 0, 1/2$ is the multiplicity label.

The Fourier components of the function $f \in L^2(S^1 \times S^1)$ corresponding to the representation of the positive and negative discrete series are given by the unitary operators $C^{l+}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}$ and $C^{l-}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}$, $l = 0, 1, 2, \ldots$, respectively

$$(C^{l\pm}_{j_1 j_2} f)(u) = \int_{S^1\times S^1} K^{\pm}(j_1 s,\, j_2 t;\, lu) f(s,t)\,\mathrm{d}s\,\mathrm{d}t \tag{3.15}$$

where

$$K^{\pm}(j_1 s,\, j_2 t;\, lu) = \frac{1}{2}\sum_{\varepsilon=0}^{1/2}[1 + (-1)^{l+2\varepsilon}\,\mathrm{e}^{\pm\mathrm{i}\pi(j_2-j_1)}]K_{\varepsilon}(j_1 s,\, j_2 t;\, lu). \tag{3.16}$$

The following relation holds

$$C^{l\pm}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(T_{\mathrm{i}\rho_1-1/2} \otimes T_{\mathrm{i}\rho_2-1/2}(g)) = T_l^{\pm}(g)C^{l\pm}_{\mathrm{i}\rho_1-1/2\,\mathrm{i}\rho_2-1/2}. \tag{3.17}$$

Furthermore, we have the Plancherel formula

$$\int_{S^1\times S^1} |f(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t = \sum_{\varepsilon=0}^{1/2}\int_0^\infty \omega(\mathrm{i}\rho - \tfrac{1}{2}, \varepsilon)\|C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2} f\|_{H_{\mathrm{i}\rho-1/2}}\,\mathrm{d}\rho$$
$$+ \sum_{l=0}^{\infty}\omega_{l,\varepsilon}\{\|C^{l+}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2} f\|_{H_l^+} + \|C^{l-}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2} f\|_{H_l^-}\} \tag{3.18}$$

where the norms come from the scalar products (2.6), (2.10) and (2.11); the Plancherel weights $\omega(\mathrm{i}\rho - \frac{1}{2}, \varepsilon)$ and $\omega_{l,\varepsilon}$ are defined by

$$\omega(\mathrm{i}\rho - \tfrac{1}{2}, \varepsilon) = (8\pi^2)^{-1}\rho th\pi\rho|\gamma(\mathrm{i}\rho_2 - \mathrm{i}\rho_1, \mathrm{i}\rho - \tfrac{1}{2}, \varepsilon)|^2 \tag{3.19}$$

and

$$\omega_{l\varepsilon} = 2^{-2l-4}\pi^{-2}(2l+1)\{\gamma(\mathrm{i}\rho_2 - \mathrm{i}\rho_1, l, \varepsilon)\gamma(\mathrm{i}\rho_2 - \mathrm{i}\rho_1, -l, \varepsilon)\}^{-1} \tag{3.20}$$

with $l = 0, 1, \ldots$, where the parameter ε in (3.20) may be any number from $\{0, 1/2\}$ if $\rho_1 \neq \rho_2$ and $\varepsilon \equiv l + 1(\mathrm{mod}2)$ only if $\rho_1 = \rho_2$; the function γ is given by (3.8).

By applying both sides of (3.14) to basis vectors (3.2) we obtain

$$\sum_{m'_1,m'_2=-\infty}^{\infty} C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m';m'_1,m'_2)t^{\mathrm{i}\rho_1-1/2}_{m'_1m_1}(g)t^{\mathrm{i}\rho_2-1/2}_{m'_2m_2}(g) = \sum_{m=-\infty}^{\infty} t^{\mathrm{i}\rho-1/2}_{m'm}(g)C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) \tag{3.21}$$

where

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) \equiv (C^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}f^{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}_{m_1m_2}, f^{\mathrm{i}\rho-1/2}_m)_{\mathrm{i}\rho-1/2} \tag{3.22}$$

are the CG coefficients for the irreducible component belonging to the principal series representations and

$$t^{\mathrm{i}\rho-1/2}_{m'm}(g) \equiv (T_{\mathrm{i}\rho-1/2}(g)f^{\mathrm{i}\rho-1/2}_{m'}, f^{\mathrm{i}\rho-1/2}_m)_{\mathrm{i}\rho-1/2}$$

are the matrix elements of the principal unitary series representations of the group G. In deriving equation (3.21), we have used the relation

$$t^{\mathrm{i}\rho_1-1/2}_{m'_1m_1}(g)t^{\mathrm{i}\rho_2-1/2}_{m'_2m_2}(g) = (T_{\mathrm{i}\rho_1-1/2}\otimes T_{\mathrm{i}\rho_2-1/2}(g)f^{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}_{m_1m_2}, f^{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}_{m'_1m'_2})_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}$$

where $(,)_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}$ is the scalar product in $L^2(S^1\times S^1)$.

Taking into account the expression for $f^{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}_{m_1m_2}$ and $f^{\mathrm{i}\rho-1/2}_m$ and that $(,)_{\mathrm{i}\rho-1/2}$ is defined by equation (2.6), the CG coefficients $c^{\mathrm{i}\rho-1/2,\varepsilon}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)$, after (3.22), can be written as

$$C^j_{j_1j_2}(m;m_1,m_2) = \frac{1}{(2\pi)^{3/2}}\left[\frac{\Gamma(-j_1+m_1)\Gamma(-j_2+m_2)\Gamma(1+j+m)}{\Gamma(1+j_1+m_1)\Gamma(1+j_2+m_2)\Gamma(-j+m)}\right]^{1/2} \times \iint_0^{2\pi}\int K_\varepsilon(j_1s, j_2t; ju)\exp(\mathrm{i}m_1\varphi_1+\mathrm{i}m_2\varphi_2-\mathrm{i}m\varphi)\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi. \tag{3.23}$$

Analogously one finds from (3.17) that

$$\sum_{m'_1,m'_2=-\infty}^{\infty} C^{l_+}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m';m'_1,m'_2)t^{\mathrm{i}\rho_1-1/2}_{m'_1m_1}(g)t^{\mathrm{i}\rho_2-1/2}_{m'_2m_2}(g) = \sum_{m=l+1}^{\infty} t^{l_+}_{m'm}(g)C^{l_+}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)$$

$$\sum_{m'_1,m'_2=-\infty}^{\infty} C^{l_-}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m';m'_1,m'_2)t^{\mathrm{i}\rho_1-1/2}_{m'_1m_1}(g)t^{\mathrm{i}\rho_2-1/2}_{m'_2m_2}(g) = \sum_{m=-\infty}^{-l-1} t^{l_-}_{m'm}(g)C^{l_-}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)$$

where

$$C^{l_+}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) = (C^{l_+}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}f^{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}_{m_1m_2}, f^{l_+}_m)_{l_+}$$

and

$$C^{l_-}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) = (C^{l_-}_{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}f^{\mathrm{i}\rho_1-1/2,\mathrm{i}\rho_2-1/2}_{m_1m_2}, f^{l_-}_m)_{l_-}$$

are CG coefficients for the irreducible component belonging to positive and negative series representations, respectively, and

$$t^{l_\pm}_{m'm}(g) \equiv (T^\pm_l(g)f^{l\pm}_m, f^{l\pm}_{m'})_{l\pm}$$

are the matrix elements of the positive (negative) series representations of G. Hence, we have the following integral representation for $c^{l\pm}_{i\rho_1-1/2,i\rho_2-1/2}(m; m_1, m_2)$

$$C^{l_+}_{j_1 j_2}(m; m_1, m_2) = \frac{1}{(2\pi)^{3/2}} \left[\frac{\Gamma(-j_1+m_1)\Gamma(-j_2+m_2)\Gamma(1+l+m)}{\Gamma(1+j_1+m_1)\Gamma(1+j_2+m_2)\Gamma(-l-m)} \right]^{1/2} \times \iint_0^{2\pi} \int K^+(j_1 s, j_2 t; lu)\, e^{im_1\varphi_1 + im_2\varphi_2 - im\varphi}\, d\varphi_1\, d\varphi_2\, d\varphi \tag{3.24}$$

$$C^{l_-}_{j_1 j_2}(m; m_1, m_2) = \frac{1}{(2\pi)^{3/2}} \left[\frac{\Gamma(-j_1+m_1)\Gamma(-j_2+m_2)\Gamma(1+l-m)}{\Gamma(1+j_1+m_1)\Gamma(1+j_2+m_2)\Gamma(-l-m)} \right]^{1/2} \times \iint_0^{2\pi} \int K^-(j_1 s, j_2 t; lu)\, e^{im_1\varphi_1 + im_2\varphi_2 - im\varphi}\, d\varphi_1\, d\varphi_2\, d\varphi. \tag{3.25}$$

Furthermore, we have the following completeness relation for CG coefficients

$$\begin{aligned}\delta_{m_1 m_1'}\delta_{m_2 m_2'} = \sum_{\varepsilon=0}^{1/2} \int_0^\infty \omega(i\rho - \tfrac{1}{2}, \varepsilon) C^{i\rho-1/2,\varepsilon}_{i\rho_1-1/2, i\rho_2-\frac{1}{2}}(m; m_1, m_2)\overline{C^{i\rho-1/2,\varepsilon}_{i\rho_1-1/2,i\rho_2-1/2}(m; m_1', m_2')} \\ + \sum_{l=0}^{\infty} \omega_{l\varepsilon} \Bigg(\sum_{m=l+1}^{\infty} C^{l_+}_{i\rho_1-1/2,i\rho_2-1/2}(m; m_1, m_2)\overline{C^{l_+}_{i\rho_1-1/2,i\rho_2-1/2}(m; m_1', m_2')} \\ + \sum_{m=-\infty}^{-j-1} C^{l_-}_{i\rho_1-1/2,i\rho_2-1/2}(m; m_1, m_2)\overline{C^{l_-}_{i\rho_1-1/2,i\rho_2-1/2}(m; m_1', m_2')}\Bigg)\end{aligned} \tag{3.26}$$

where $\omega(i\rho - 1/2, \varepsilon)$ and $\omega_{l\varepsilon}$ are given by (3.19) and (3.20), respectively.

Let us calculate the CG coefficient for the case of three principal series. In order to compute integrals in (3.23) we use the Fourier expansion

$$\left|\sin\frac{\varphi}{2}\right|^{-2a} \text{sign}^{2\varepsilon} \sin\frac{\varphi}{2} = \sum_{n=-\infty}^{\infty} A_n\, e^{i(n+\varepsilon)\varphi} \qquad 0 \leqslant \varphi < 2\pi \tag{3.27}$$

where

$$A_n = \frac{1}{2}\int_0^{2\pi} \left|\sin\frac{\varphi}{2}\right|^{-2a} \text{sign}^{2\varepsilon} \sin\frac{\varphi}{2}\, e^{-i(n+\varepsilon)\varphi}\, d\varphi = \frac{1}{\sqrt{\pi}} \times e^{-i\pi\varepsilon} \frac{\Gamma(\frac{1}{2}-a+\varepsilon)}{\Gamma(a+\varepsilon)} \frac{\Gamma(a+n+\varepsilon)}{\Gamma(1-a+n+\varepsilon)} \tag{3.28}$$

(see formulae (3.631.1) and (3.631.8) from [14]). Hence, it follows that

$$\begin{aligned}C^{j,\varepsilon}_{j_1 j_2}(m; m_1, m_2) = 2^{-1+j-(j_1+j_2)/2}\gamma(j_2 - j_1, j, \varepsilon) \\ \times e^{i\pi\varepsilon} \frac{\Gamma(\frac{1}{2}-a_1+\varepsilon)\Gamma(\frac{1}{2}-a_2+\varepsilon)\Gamma(\frac{1}{2}-a_3+\varepsilon)}{\Gamma(a_1+\varepsilon)\Gamma(a_2+\varepsilon)\Gamma(a_3+\varepsilon)} \\ \times \left[\frac{\Gamma(-j_1+m_1)\Gamma(-j_2+m_2)\Gamma(1+j+m)}{\Gamma(1+j_1+m_1)\Gamma(1+j_2+m_2)\Gamma(-j-m)}\right]^{1/2} S\end{aligned} \tag{3.29}$$

where

$$S = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a_1+m_1+\varepsilon+n)\Gamma(a_2-m_2+\varepsilon+n)\Gamma(a_3+\varepsilon+n)}{\Gamma(1-a_1+m_1+\varepsilon+n)\Gamma(1-a_2-m_2+\varepsilon+n)\Gamma(1-a_3+\varepsilon+n)}. \tag{3.30}$$

Table 1. The expressions for α and β in terms of j_1, j_2, j, m_1, m_2, and m.

$\alpha_{012} = -j_2 + m_2$	$\alpha_{024} = -1 - j_1 - j_2 - j$	$\alpha_{123} = 1 + j + m$	$\alpha_{145} = 1 + j_1 + j_2 - j$
$\alpha_{013} = 1 + j_1 - j_2 + j$	$\alpha_{025} = -j_1 - m_1$	$\alpha_{124} = -j + m$	$\alpha_{234} = -j_1 + m_1$
$\alpha_{014} = j_1 - j_2 - j$	$\alpha_{034} = -j_2 - m_2$	$\alpha_{125} = 1 + j_2 + m_2$	$\alpha_{235} = 1 - j_1 + j_2 + j$
$\alpha_{015} = 1 + j_1 - m_1$	$\alpha_{035} = 1 + j - m$	$\alpha_{134} = 1 + j_1 + m_1$	$\alpha_{245} = -j_1 + j_2 - j$
$\alpha_{023} = -j_1 - j_2 + j$	$\alpha_{045} = -j - m$	$\alpha_{135} = 2 + j_1 + j_2 + j$	$\alpha_{345} = 1 + j_2 - m_2$
$\beta_{01} = -j_1 - j_2 - m$	$\beta_{05} = -2j_2$	$\beta_{15} = 1 + j_1 - j_2 + m$	$\beta_{34} = 2 + 2j$
$\beta_{02} = 1 + j_1 - j_2 - m$	$\beta_{12} = 2 + 2j_1$	$\beta_{23} = -j_1 - j + m_2$	$\beta_{35} = 1 - j_2 + j + m_1$
$\beta_{03} = -j_2 - j - m_1$	$\beta_{13} = 1 + j_1 - j + m_2$	$\beta_{24} = 1 - j_1 + j + m_2$	$\beta_{45} = -j_2 - j + m_1$
$\beta_{04} = 1 - j_2 + j - m_1$	$\beta_{14} = 2 + j_1 + j + m_2$	$\beta_{25} = -j_1 - j_2 + m_2$	

Calculation of the sum (3.30) proceeds as in the previous paper [7] (see appendix A). As a result the sum (3.30) is expressed in terms of the generalized hypergeometric function ${}_3F_2$ with unit argument

$$S = \pi \left\{ \cot\pi\left(\frac{j_1 - j_2 - j}{2} + m_1 + \varepsilon\right) \frac{\Gamma(-j_1 + j_2 - m)\Gamma(1 + j_2 + j - m_1)}{\Gamma(-j_1 + j_2 + j)\Gamma(1 + j - m)\Gamma(-j_1 - m_1)} \right.$$
$$\times {}_3F_2\left[\begin{matrix} j_1 - j_2 - j, -j + m, 1 + j_1 + m_1; \\ 1 + j_1 - j_2 + m, -j_2 - j + m_1; \end{matrix}\; 1\right]$$
$$+ \cot\pi\left(\frac{-j_1 + j_2 - j}{2} - m_2 + \varepsilon\right)$$
$$\times \frac{\Gamma(j_1 - j_2 + m)\Gamma(1 + j_1 + j + m_2)}{\Gamma(1 + j_1 - j_2 + j)\Gamma(1 + j + m)\Gamma(-j_2 + m_2)}$$
$$\times {}_3F_2\left[\begin{matrix} -j_1 + j_2 - j, -j - m_2, 1 + j_2 - m_2; \\ 1 - j_1 + j_2 + m, -j_1 - j - m_2; \end{matrix}\; 1\right]$$
$$+ \cot\pi\left(\frac{2 + j_1 + j_2 + j}{2} + \varepsilon\right)$$
$$\times \frac{\Gamma(-1 - j_2 - j + m_1)\Gamma(-1 - j_1 - j - m_2)}{\Gamma(-1 - j_1 - j_2 - j)\Gamma(-j_2 - m_2)\Gamma(-j_1 + m_1)}$$
$$\left. \times {}_3F_2\left[\begin{matrix} 2 + j_1 + j_2 + j, 1 + j_2 + m_2, 1 + j_1 - m_1; \\ 2 + j_2 + j - m_1, 2 + j_1 + j + m_2; \end{matrix}\; 1\right]\right\}. \tag{3.31}$$

There are two-term and three-term relations between the series ${}_3F_2(1)$. These relations were derived by Thomae and are investigated in more familiar notation by Whipple [15] (see appendix B). For our purpose we express the Whipple parameters r_i, $i = 0, 1, \ldots, 5$, in terms of j_1, j_2, j, m_1, m_2 and m as in [5]

$$3r_0 = -\tfrac{3}{2} - 3j_2 - m_1 - m \qquad 3r_1 = \tfrac{3}{2} + 3j_1 + m_2 + m \qquad 3r_2 = -\tfrac{3}{2} - 3j_1 + m_2 + m$$
$$3r_3 = \tfrac{3}{2} + 3j + m_1 - m_2 \qquad 3r_4 = -\tfrac{3}{2} - 3j + m_1 - m_2 \qquad 3r_5 = \tfrac{3}{2} + 3j_2 - m_1 - m. \tag{3.32}$$

In table 1 the relationships between the set $(\alpha_{lmn}, \beta_{mn})$ and the set $(j_1, j_2, j, m_1, m_2, m)$ are given explicitly.

As a result the sum S can be rewritten in terms of the Whipple functions

$$S = \pi^3\Gamma(\alpha_{023})\left\{\frac{\cot\pi((\alpha_{135}/2)-\beta_{54}+\varepsilon)}{\sin\pi\beta_{15}\sin\pi\beta_{45}}\frac{F_p(5;1,4)}{\Gamma(\alpha_{235})\Gamma(\alpha_{035})\Gamma(\alpha_{025})}\right.$$
$$+\frac{\cot\pi((\alpha_{135}/2)-\beta_{14}+\varepsilon)}{\sin\pi\beta_{51}\sin\pi\beta_{41}}\frac{F_p(1;4,5)}{\Gamma(\alpha_{023})\Gamma(\alpha_{123})\Gamma(\alpha_{012})}$$
$$\left.+\frac{\cot\pi((\alpha_{135}/2)+\varepsilon)}{\sin\pi\beta_{54}\sin\pi\beta_{14}}\frac{F_p(4;1,5)}{\Gamma(\alpha_{024})\Gamma(\alpha_{034})\Gamma(\alpha_{234})}\right\}. \tag{3.33}$$

By using three-term relations from appendix B the CG coefficients can be expressed in terms of two hypergeometric functions ${}_3F_2(1)$ with unit argument. For example, by virtue of equality (B.9) (see appendix B)

$$\frac{\sin\pi\beta_{14}F_p(5)}{\Gamma(\alpha_{235})\Gamma(\alpha_{035})\Gamma(\alpha_{025})}+\frac{\sin\pi\beta_{45}F_p(1)}{\Gamma(\alpha_{013})\Gamma(\alpha_{123})\Gamma(\alpha_{012})}+\frac{\sin\pi\beta_{51}F_p(4)}{\Gamma(\alpha_{024})\Gamma(\alpha_{034})\Gamma(\alpha_{234})}=0 \tag{3.34}$$

the CG coefficient can be written in the form

$$C^{j\varepsilon}_{j_1j_2}(m;m_1,m_2)=\delta_{m,m_1+m_2}\,\mathrm{e}^{\mathrm{i}\pi(m-\varepsilon)}(2\pi)^{3/2}\gamma 2^{1+(j_1+j_2)/2}\left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{125})\Gamma(\alpha_{124})}\right]^{1/2}$$
$$\times\frac{\Gamma(\alpha_{235})\Gamma(\alpha_{023})}{\sin\pi\beta_{14}}\left\{\sin\pi(\beta_{14}-(\alpha_{135}/2)+\varepsilon)\frac{\Gamma(\alpha_{013})F_p(4)}{\Gamma(\alpha_{034})\Gamma(\alpha_{234})}\right.$$
$$\left.+\sin\pi((\alpha_{135}/2)+\varepsilon)\frac{\Gamma(\alpha_{024})F_p(1)}{\Gamma(\alpha_{123})\Gamma(\alpha_{012})}\right\}. \tag{3.35}$$

By using the relations from appendix B one can find a large number of other expressions for the CG coefficients in terms of ${}_3F_2(1)$.

It is also worth noting that the formula (3.30) defines the most symmetric expression for the CG coefficient and it can be written in terms of one special function, namely in terms of the bilateral series ${}_3H_3$ with unit argument [15] (see appendix C)

$$S=\frac{\Gamma(1-a_1+m_1+\varepsilon)\Gamma(1-a_2-m_2+\varepsilon)\Gamma(1-a+\varepsilon)}{\Gamma(a_1+m_1+\varepsilon)\Gamma(a_2-m_2+\varepsilon)\Gamma(a+\varepsilon)}$$
$$\times{}_3H_3\left[\begin{matrix}a_1+m_1+\varepsilon,a_2-m_2+\varepsilon,a+\varepsilon;\\ 1-a_1+m_1+\varepsilon,1-a_2-m_2+\varepsilon,1-a+\varepsilon;\end{matrix}\;1\right]. \tag{3.36}$$

It is evident that the series ${}_3H_3\left[\begin{matrix}c_1,c_2,c_3;\\ d_1,d_2,d_3;\end{matrix}\;1\right]$ is not changed under a permutation of $(c_1c_2c_3)$ or $(d_1d_2d_3)$. This property of ${}_3H_3(1)$ implies $3!\times 3!=36$ symmetry relations for CG coefficients which also include (formally) Regge-type symmetry relations. For example, the replacements

$$\begin{aligned}&j_1\to\frac{j_1+j_2-m}{2} &&m_1\to\frac{-j_1+j_2+m_1-m_2}{2}\\ &j_2\to\frac{j_1+j_2+m}{2} &&m_2\to\frac{-j_1+j_2-m_1+m_2}{2}\\ &j_1\to j\end{aligned} \tag{3.37}$$

correspond to

$$\begin{aligned}&a_1+m_1\to a_2-m_2 &&1-a_1+m_1\to 1-a_1-m_1\\ &a_2-m_2\to a_1+m_1 &&1-a_2-m_2\to 1-a_2-m_2\\ &a\to a.\end{aligned} \tag{3.38}$$

A complete discussion of the symmetry properties of CG coefficients is not attempted here.

The expression for CG coefficients, which couples two principal series into a representation of the positive (negative) discrete representation, are analysed in the same way. We have

$$C^{l_+}_{j_1 j_2}(m; m_1, m_2) = \delta_{m,m_1+m_2}\, e^{i\pi(m_1+(\alpha_{245}/2))}(2\pi)^{3/2}2^{1+(j_1+j_2)/2}\left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{123})\Gamma(\alpha_{124})}\right]^{1/2} \times \frac{\Gamma(\alpha_{235})\Gamma(\alpha_{024})\Gamma(\alpha_{023})}{\Gamma(\alpha_{123})\Gamma(\alpha_{012})} F_p(1) \tag{3.39}$$

with $l = 0, 1, 2, \ldots, m = l+1, l+2, \ldots$ and

$$C^{l_-}_{j_1 j_2}(m; m_1, m_2) = \delta_{m,m_1+m_2}\, e^{i\pi(m_2+(\alpha_{014}/2))}(2\pi)^{3/2}2^{1+(j_1+j_2)/2}\left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{035})}{\Gamma(\alpha_{134})\Gamma(\alpha_{125})\Gamma(\alpha_{045})}\right]^{1/2} \times \frac{\Gamma(\alpha_{013})\Gamma(\alpha_{024})\Gamma(\alpha_{023})}{\Gamma(\alpha_{035})\Gamma(\alpha_{025})} F_p(5) \tag{3.40}$$

with $l = 0, 1, 2, \ldots, m = -l-1, -l-2, \ldots$.

In deriving (3.39) and (3.40) we have used the fact that the three-term relations between ${}_3F_2(1)$ functions reduce to two-term relations when one of the parameters α_{lmn} is a negative integer or zero, namely $\alpha_{035} = 1 + l - m \leqslant 0$ ($\alpha_{123} = 1 + l + m \leqslant 0$).

4. The tensor product of a complementary series representation with a representation of the principal series

Let $H_{\tau_1, i\rho_2-1/2}$ be the Hilbert space completion of $C^\infty(S^1 \times S^1)$ with respect to the norm defined in terms of the scalar product [10, 11]

$$(f_1, f_2)_{\tau_1, i\rho_2-1/2} = \frac{2^{\tau_1}}{\sqrt{\pi}} \frac{\Gamma(\tau_1+1)}{\Gamma(-(2\tau_1+1)/2)} \int_{S^1\times S^1\times S^1} [s_1, s_2]^{-1-\tau_1} f_1(s_1, t)\overline{f_2(s_2, t)}\, ds_1\, ds_2\, dt \tag{4.1}$$

where $f_1, f_2 \in C^\infty(S^1 \times S^1)$ and [,] is given by the formula (2.1). The functions $f^{\tau_1, i\rho_2-1/2}_{m_1 m_2}$

$$f^{\tau_1, i\rho_2-1/2}_{m_1 m_2}(s, t) = \frac{1}{2\pi}\left[\frac{\Gamma(-\tau_1+m_1)\Gamma(\frac{1}{2}-i\rho_2+m_2)}{\Gamma(1+\tau_1+m_1)\Gamma(\frac{1}{2}+i\rho_2+m_2)}\right]^{1/2} e^{im_1\varphi_1+im_2\varphi_2} \tag{4.2}$$

with $s = (\cos\varphi_1, \sin\varphi_1, 1)$ and $t = (\cos\varphi_2, \sin\varphi_2, 1)$, form the orthonormal basis in $H_{\tau_1, i\rho_2-1/2}$. The tensor product $T_{\tau_1} \otimes T_{i\rho_2-1/2}$ of a complementary series representation T_{τ_1}, $-1 < \tau_1 < -1/2$, with a representation $T_{i\rho_2-1/2}$, $\rho_2 \geqslant 0$, of the principal series can be realized in the Hilbert space $H_{\tau_1, i\rho_2-1/2}$. At $j_1 = \tau_1$ and $j_2 = -1/2 + i\rho_2$ the formula (3.3) gives the representation operator in this case.

Let $\mathsf{F}_{\tau_1, i\rho_2-1/2}$ be the Hilbert space completion of $K(X)$ with respect to the scalar product

$$(\Phi_1, \Phi_2)_{\tau_1, i\rho_2-1/2} = \int_X \Phi_1(x) B_{\tau_1, i\rho_2-1/2}(U_{i\rho_2-\tau_1-1/2}(\alpha_x)\Phi_2)\, dx$$

where $\Phi_1, \Phi_2 \in K(X)$ and $B_{\tau_1, i\rho_2-1/2}$ is the generalized function on X

$$B_{\tau_1, i\rho_2-1/2}(\Phi) = \frac{2^{\tau_1}}{\sqrt{\pi}} \frac{\Gamma(\tau_1+1)}{\Gamma(-(2\tau_1+1)/2)} \int_{-\infty}^{\infty} (q^2+1)^{(2\tau_1+2i\rho_2+1)/4}|q|^{2\tau_1-2}\Phi(1, q, q)\, dq.$$

The operator $P_{\tau_1, i\rho_2-1/2}$ establishes the unitary equivalence between $T_{\tau_1} \otimes T_{i\rho_2-1/2}$ and a unitary representation of G acting in the space $\mathsf{F}_{\tau_1, i\rho_2-1/2}$, which is obtained by extension of the representation $U_{i\rho_2-\tau_1-1/2}$.

Furthermore, the intertwining mapping given by (3.7) induces a unitary operator from $\mathsf{F}_{\tau_1,\mathrm{i}\rho_2-1/2}$ into a direct integral of the carrier spaces of $T_{\mathrm{i}\rho-1/2}$ and $T_l^+ \oplus T_l^-$. Thus, the tensor product $T_{\tau_i} \otimes T_{\mathrm{i}\rho_2-1/2}$ contains two copies of a direct integral plus a direct sum of discrete series representations

$$T_{\tau_1} \otimes T_{\mathrm{i}\rho_2-1/2} = 2\int_0^\infty T_{-1/2+\mathrm{i}\rho}\,\mathrm{d}\rho \oplus \sum_{l=0}^{\infty}(T_l^+ \oplus T_l^-). \tag{4.3}$$

At $j_1 = \tau_1$ and $j_2 = -1/2 + \mathrm{i}\rho_2$ the formulae (3.10) and (3.15) give this decomposition and the following relations hold

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2}(T_{\tau_1} \otimes T_{\mathrm{i}\rho_2-1/2})(g) = T_{\mathrm{i}\rho-1/2}(g)C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2} \tag{4.4}$$

$$C^{l\pm}_{\tau_1,\mathrm{i}\rho_2-1/2}(T_{\tau_1} \otimes T_{\mathrm{i}\rho_2-1/2})(g) = T_l^{\pm}(g)C^{l\pm}_{\tau_1,\mathrm{i}\rho_2-1/2}. \tag{4.5}$$

The corresponding Plancherel formula is given by

$$\begin{aligned}\frac{2^{\tau_1}}{\sqrt{\pi}}\frac{\Gamma(\tau_1+1)}{\Gamma(-(2\tau_1+1)/2)}&\int_{S^1\times S^1\times S^1}[s_1,s_2]^{-1-\tau_1} f_1(s_1,t)\overline{f_2(s_2,t)}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t\\ &= \int_0^\infty \sum_{\varepsilon=0}^{1/2}\omega(-\tfrac{1}{2}+\mathrm{i}\rho,\varepsilon)\|C^{-1/2+\mathrm{i}\rho,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2}f\|_{H_{\mathrm{i}\rho-1/2}}\,\mathrm{d}\rho\\ &\quad+\sum_{l=0}^{\infty}\omega_{l,\varepsilon}\{\|C^{l_+}_{\tau_1,\mathrm{i}\rho_2-1/2}f\|_{H_l^+} + \|C^{l_-}_{\tau_1,\mathrm{i}\rho_2-1/2}f\|_{H_l^-}\}.\end{aligned} \tag{4.6}$$

Here $\omega(\mathrm{i}\rho - 1/2, \varepsilon)$ and $\omega_{l,\varepsilon}$ are defined by

$$\omega(\mathrm{i}\rho-1/2,\varepsilon) = 2^{\tau_1-7/2}\pi^{-3/2}\rho t h\pi\rho\Delta(\tau_1,\mathrm{i}\rho_2-1/2,\mathrm{i}\rho-1/2) \tag{4.7}$$

and

$$\omega_{l,\varepsilon} = 2^{-7/2-l}\pi^{-2}l!(2l+1)\frac{1}{\Gamma(1+j_1)}\Delta(\tau_1,\mathrm{i}\rho_2-1/2,l) \tag{4.8}$$

where

$$\begin{aligned}\Delta(j_1,j_2,j) = {}&\Gamma\left(\frac{j_1+j_2-j+2\varepsilon+1}{2}\right)\Gamma\left(\frac{j_1-j_2-j+2\varepsilon}{2}\right)\\ &\times\Gamma\left(\frac{j_1+j_2+j+2\varepsilon+1}{2}\right)\Gamma\left(\frac{j_1-j_2+j+2\varepsilon+1}{2}\right).\end{aligned} \tag{4.9}$$

(In equation (4.8) the parameter ε may be any number from $\{0, 1/2\}$.)

It follows from (4.4) and (4.5) that the following relations between CG coefficients and the matrix elements of UIRs hold

$$\begin{aligned}\sum_{m_1',m_2'=-\infty}^{\infty} &C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2}(m';m_1',m_2')t^{\tau_1}_{m_1'm_1}(g)t^{\mathrm{i}\rho_2-1/2}_{m_2',m_2}(g)\\ &= \sum_{m=-\infty}^{\infty} t^{\mathrm{i}\rho-1/2}_{m'm}(g)C^{\mathrm{i}\rho-1/2}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)\end{aligned} \tag{4.10}$$

$$\begin{aligned}\sum_{m_1',m_2'=-\infty}^{\infty} &C^{l_+}_{\tau_1,\mathrm{i}\rho_2-1/2}(m';m_1',m_2')t^{\tau_1}_{m_1'm_1}(g)t^{\mathrm{i}\rho_2-1/2}_{m_2'm_2}(g)\\ &= \sum_{m=l+1}^{\infty} t^{l_+}_{m'm}(g)C^{l_+}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)\end{aligned} \tag{4.11}$$

$$\sum_{m'_1,m'_2=-\infty}^{\infty} C^{l_-}_{\tau_1,\mathrm{i}\rho_2-1/2}(m';m'_1,m'_2)t^{\tau_1}_{m'_1m_1}(g)t^{\mathrm{i}\rho_2-1/2}_{m'_2m_2}(g)$$

$$= \sum_{m=-\infty}^{-l-1} t^{l_-}_{m'm}(g)C^{l_-}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) \tag{4.12}$$

where

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) \equiv (C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2}f^{\tau_1,\mathrm{i}\rho_2-1/2}_{m_1m_2},\, f^{\mathrm{i}\rho-1/2}_{m})_{\mathrm{i}\rho-1/2} \tag{4.13}$$

$$C^{l\pm}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) \equiv (C^{l\pm}_{\tau_1,\mathrm{i}\rho_2-1/2}f^{\tau_1,\mathrm{i}\rho_2-1/2}_{m_1m_2},\, f^{l\pm}_{m})_{l\pm} \tag{4.14}$$

are the CG coefficients of the tensor product $T_{\tau_1} \otimes T_{\mathrm{i}\rho_2-1/2}$.

Furthermore, we have the following completeness relation

$$\int_0^\infty \sum_{\varepsilon=0}^{1/2}\sum_{m=-\infty}^{\infty} \omega(\mathrm{i}\rho-1/2,\varepsilon)C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m'_1,m'_2)}$$
$$+\sum_{l=0}^{\infty}\omega_{l,\varepsilon}\left(\sum_{m=l+1}^{\infty} C^{l_+}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)\overline{C^{l_+}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m'_1,m'_2)}\right.$$
$$\left.+\sum_{m=-\infty}^{-l-1} C^{l_-}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m_1,m_2)\overline{C^{l_-}_{\tau_1,\mathrm{i}\rho_2-1/2}(m;m'_1,m'_2)}\right) = \delta_{m_1m'_1}\delta_{m_2m'_2} \tag{4.15}$$

where $\omega(\mathrm{i}\rho-1/2,\varepsilon)$ and $\omega_{l,\varepsilon}$ are given by (4.7) and (4.8), respectively.

At $j_1 = \tau_1, -1 < \tau_1 < -1/2$, and $j_2 = -1/2 + \mathrm{i}\rho_2$ formulae (3.23), (3.24) and (3.25) ((3.35), (3.39) and (3.40)) give corresponding CG coefficients.

5. The tensor product of complementary series representations

The tensor product $T_{\tau_1} \otimes T_{\tau_2}$, $-1 < \tau_{\mathrm{i}} < -1/2$, $i = 1, 2$ of complementary series representations act on $H_{\tau_1\tau_2}$ isomorphic to the Hilbert space completion of $C^\infty(S^1 \times S^1)$ with respect to the scalar product

$$(f_1, f_2)_{\tau_1\tau_2} = \frac{2^{\tau_1+\tau_2}}{\pi}\frac{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}{\Gamma(-(2\tau_1+1)/2)\Gamma(-(2\tau_2+1)/2)}$$
$$\times\int_{S^1\times S^1\times S^1\times S^1} |[s_1,s_2]|^{-1-\tau_1}|[t_1,t_2]|^{-1-\tau_2} f_1(s_1,t_1)$$
$$\times\overline{f_2(s_2,t_2)}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{5.1}$$

where $f \in C^\infty(S^1 \times S^2)$ and [,] is given by (2.1).

The functions $f^{\tau_1\tau_2}_{m_1m_2}$ are

$$f^{\tau_1\tau_2}_{m_1m_2}(s,t) = \frac{1}{2\pi}\left[\frac{\Gamma(-\tau_1+m_1)\Gamma(-\tau_2+m_2)}{\Gamma(1+\tau_1+m_1)\Gamma(1+\tau_2+m_1)}\right]^{1/2} \mathrm{e}^{\mathrm{i}m_1\varphi_1+\mathrm{i}m_2\varphi_2} \tag{5.2}$$

with $s = (\cos\varphi_1, \sin\varphi_1, 1)$ and $t = (\cos\varphi_2, \sin\varphi_2, 1)$, form the orthonormal basis in $H_{\tau_1\tau_2}$. At $j_i = \tau_1$, $i = 1, 2$, the formula (3.3) gives the representation operator.

Let $\mathsf{F}_{\tau_1\tau_2}$ be the Hilbert space completion of $K(X)$ with respect to the scalar product

$$(\Phi_1, \Phi_2)_{\tau_1\tau_2} = \int_X \Phi_1(x)B_{\tau_1\tau_2}(U_{\tau_2-\tau_1}(\alpha_x)\Phi_2)\,\mathrm{d}x$$

where $\Phi_1, \Phi_2 \in K(X)$ and B_{τ_1,τ_2} is the generalized function on X

$$B_{\tau_1,\tau_2}(\Phi) = \frac{2^{2\tau_1+2\tau_2+2}}{\pi}\frac{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}{\Gamma(-(2\tau_1+1)/2)\Gamma(-(2\tau_2+1)/2)} \times \int_X |[s^-, s^-\alpha_x]|^{-\tau_1-1}|[s^+, s^+\alpha_x]|^{-\tau_2-1}\Phi(x)\,\mathrm{d}x.$$

The operator $P_{\tau_1\tau_2}$ establishes the unitary equivalence between $T_{\tau_1}\otimes T_{\tau_2}$ and a unitary representation of G acting in the space $\mathsf{F}_{\tau_1,\tau_2}$, which is obtained by extension of the representation $U_{\tau_2-\tau_1}$.

The mapping given by (3.7) induces a unitary operator from $\mathsf{F}_{\tau_1,\tau_2}$ into a direct integral of the carrier spaces of $T_{\mathrm{i}\rho-1/2}$ and $T_l^+\oplus T_l^-$ when $\tau_1+\tau_2 > -3/2$ and $T_{\mathrm{i}\rho-1/2}$, $T_l^+\oplus T_l^-$ and $T_{\tau_1+\tau_2+1}$, when $\tau_1+\tau_2 < -3/2$. Thus, the structure of CG series have the form

(i)

$$T_{\tau_1}\otimes T_{\tau_2} = 2\int_0^\infty T_{-1/2+\mathrm{i}\rho}\,\mathrm{d}\rho \oplus \sum_{l=0}^{\infty}(T_l^+\oplus T_l^-) \qquad \text{when } \tau_1+\tau_2 > -3/2 \tag{5.3}$$

(ii)

$$T_{\tau_1}\otimes T_{\tau_2} = 2\int_0^\infty T_{-1/2+\mathrm{i}\rho}\,\mathrm{d}\rho \oplus \sum_{l=0}^{\infty}(T_l^+\oplus T_l^-)\oplus T_{\tau_1+\tau_2+1} \qquad \text{when } \tau_1+\tau_2 < -3/2. \tag{5.4}$$

In case (ii) $H_{\tau_1\tau_2}$ contains a sub-space which is isomorphic to one copy of the complementary series representation $T_{\tau_1+\tau_2+1}$.

In both cases formulae (3.10) and (3.15) at

$$j_i = \tau_i \qquad j_2 = \tau_2 \tag{5.5}$$

define the Fourier components corresponding to the representations of the principal and discrete series, respectively. For the Fourier component transformed according to the complementary series representation $T_{\tau_1+\tau_2+1}$ it is also necessary to put

$$j = \tau_1+\tau_2+1 \qquad \varepsilon = 0 \tag{5.6}$$

in formula (3.10). We have the equalities

$$C_{\tau_1\tau_2}^{\mathrm{i}\rho-1/2,\varepsilon}(T_{\tau_1}\otimes T_{\tau_2})(g) = T_{\mathrm{i}\rho-1/2}(g)C_{\tau_1\tau_2}^{\mathrm{i}\rho-1/2,\varepsilon} \tag{5.7}$$

$$C_{\tau_1\tau_2}^{l\pm}(T_{\tau_1}\otimes T_{\tau_2})(g) = T_l^{l\pm}(g)C_{\tau_1\tau_2}^{l\pm} \tag{5.8}$$

$$C_{\tau_1\tau_2}^{\tau_1+\tau_2+1,0}(T_{\tau_1}\otimes T_{\tau_2})(g) = T_{\tau_1+\tau_2+1}(g)C_{\tau_1\tau_2}^{\tau_1+\tau_2+1,0}. \tag{5.9}$$

The Plancherel formula for the tensor product of two complementary series representations is defined by

$$\begin{aligned}\frac{2^{\tau_1+\tau_2}}{\pi}&\frac{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}{\Gamma(-(2\tau_1+1)/2)\Gamma(-(2\tau_2+1)/2)} \\ &\times\int_{S^1\times S^1\times S^1\times S^1}[s_1,s_2]^{-1-\tau_1}[t_1,t_2]^{-1-\tau_2}f_1(s_1,t_1)\overline{f_2(s_2,t_2)}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \int_0^\infty\sum_{\varepsilon=0}^{1/2}\omega(\mathrm{i}\rho-1/2,\varepsilon)\|C_{\tau_1\tau_2}^{\mathrm{i}\rho-1/2,\varepsilon}f\|_{H_{\mathrm{i}\rho-1/2}}\,\mathrm{d}\rho \\ &\quad+\sum_{l=0}^{\infty}\omega_{l,\varepsilon}\{\|C_{\tau_1\tau_2}^{l_+}f\|_{H_l^+}+\|C_{\tau_1\tau_2}^{l_-}f\|_{H_l^-}\}+\omega\|C_{\tau_1\tau_2}^{\tau_1+\tau_2+1,0}f\|_{H_{\tau_1+\tau_2+1}}\end{aligned} \tag{5.10}$$

where the norms correspond to the inner product in the representation spaces of the principal, discrete and complementary series; the functions $\omega(\mathrm{i}\rho-1/2,\varepsilon)$, $\omega_{l,\varepsilon}$, ω are given by

$$\omega(\mathrm{i}\rho-1/2,\varepsilon)=2^{-\tau_1-\tau_2-5}\pi^{-3}\tanh\pi\rho[\cosh\pi\rho+(-1)^{2\varepsilon}\sin\pi(\tau_1+\tau_2)] \times|\Gamma(\tfrac{3}{2}+\tau_1+\tau_2+\mathrm{i}\rho)|^2|\gamma(\tau_2-\tau_1,-\tfrac{1}{2}+\mathrm{i}\rho,\varepsilon)|^{-2} \quad (5.11)$$

$$\omega_{l,\varepsilon}=2^{-\tau_1-\tau_2-2l-5}\pi^{-2}(-1)^{2\varepsilon+l+1}(2l+1)\Gamma(2+l+\tau_1+\tau_2) \times\{\Gamma(l-\tau_1-\tau_2)\gamma(\tau_2-\tau_1,l,\varepsilon)\gamma(\tau_2-\tau_1,-l-1,\varepsilon)\}^{-1} \quad (5.12)$$

with $l=0,1,2,\ldots$ and ε may be any number from $\{0,1/2\}$, if $\tau_1\neq\tau_2$ and $\varepsilon=l+1(\mathrm{mod}2)$ only if $\tau_1=\tau_2$

$$\omega=\begin{cases}0 & \text{if } -\frac{3}{2}\leqslant\tau_1+\tau_2<-1\\ 2^{-\tau_1-\tau_2-2}\pi^{-3/2}\Gamma(-\frac{1}{2}-\tau_1)\Gamma(-\frac{1}{2}-\tau_2)\Gamma(\tau_1+1) & \\ \quad\times\Gamma(\tau_2+1)\Gamma(2+\tau_1+\tau_2)/\Gamma(-\frac{3}{2}-\tau_1-\tau_2) & \text{if } \tau_1+\tau_2<-\frac{3}{2}.\end{cases} \quad (5.13)$$

One can easily derive the following relations between the CG coefficients and the matrix elements of UIRs

$$\sum_{m_1'm_2'=-\infty}^{\infty}C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1\tau_2}(m';m_1',m_2')t^{\tau_1}_{m_1'm_1}(g)t^{\tau_2}_{m_2'm_2}(g)=\sum_{m=-\infty}^{\infty}t^{\mathrm{i}\rho-1/2}_{m'm}(g)C^{\mathrm{i}\rho-1/2}_{\tau_1,\tau_2}(m;m_1,m_2) \quad (5.14)$$

$$\sum_{m_1'm_2'=-\infty}^{\infty}C^{l_+}_{\tau_1\tau_2}(m';m_1',m_2')t^{\tau_1}_{m_1'm_1}(g)t^{\tau_2}_{m_2'm_2}(g)=\sum_{m=l-1}^{\infty}t^{l_+}_{m'm}(g)C^{l_+}_{\tau_1\tau_2}(m;m_1,m_2) \quad (5.15)$$

$$\sum_{m_1'm_2'=-\infty}^{\infty}C^{l_-}_{\tau_1\tau_2}(m';m_1',m_2')t^{\tau_1}_{m_1'm_1}(g)t^{\tau_2}_{m_2'm_2}(g)=\sum_{m=-\infty}^{-l-1}t^{l_-}_{m'm}(g)C^{l_-}_{\tau_1\tau_2}(m;m_1,m_2) \quad (5.16)$$

$$\sum_{m_1'm_2'=-\infty}^{\infty}C^{\tau_1+\tau_2+1}_{\tau_1\tau_2}(m';m_1',m_2')t^{\tau_1}_{m_1'm_1}(g)t^{\tau_2}_{m_2'm_2}(g)=\sum_{m=-\infty}^{\infty}t^{\tau_1+\tau_2+1}_{m'm}(g)C^{\tau_1+\tau_2+1}_{\tau_1\tau_2}(m;m_1,m_2) \quad (5.17)$$

where

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1\tau_2}(m;m_1,m_2)=(C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1\tau_2}f^{\tau_1\tau_2}_{m_1m_2},f^{\mathrm{i}\rho_2-1/2}_{m})_{\mathrm{i}\rho-1/2} \quad (5.18)$$

$$C^{l\pm}_{\tau_1\tau_2}(m;m_1,m_2)=(C^{l\pm}_{\tau_1\tau_2}f^{\tau_1\tau_2}_{m_1m_2},f^{l\pm}_{m})_{l\pm} \quad (5.19)$$

$$C^{\tau_1+\tau_2+1}_{\tau_1\tau_2}(m;m_1,m_2)=(C^{\tau_1+\tau_2+1,0}_{\tau_1\tau_2}f^{\tau_1\tau_2}_{m_1m_2},f^{\tau_1+\tau_2+1}_{m})_{\tau_1+\tau_2+1} \quad (5.20)$$

are the CG coefficients for the tensor product $T_{\tau_1}\otimes T_{\tau_2}$.

It follows from equation (5.8) that

$$\int_0^{\infty}\sum_{\varepsilon=0}^{1/2}\sum_{m=-\infty}^{\infty}\omega(\mathrm{i}\rho-1/2)C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1\tau_2}(m;m_1,m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1\tau_2}(m;m_1,m_2)} +\sum_{l=0}^{\infty}\omega_{l,\varepsilon}\left(\sum_{m=l+1}^{\infty}C^{l_+}_{\tau_1\tau_2}(m;m_1,m_2)\overline{C^{l_+}_{\tau_1\tau_2}(m;m_1,m_2)}\right) +\sum_{m=-\infty}^{-l-1}C^{l_-}_{\tau_1\tau_2}(m;m_1,m_2)\overline{C^{l_-}_{\tau_1\tau_2}(m;m_1,m_2)} +\omega\sum_{m=-\infty}^{\infty}C^{\tau_1+\tau_2+1}_{\tau_1\tau_2}(m;m_1,m_2)\times\overline{C^{\tau_1+\tau_2+1}_{\tau_1\tau_2}(m;m_1',m_2')}=\delta_{m_1m_1'}\delta_{m_2m_2'}. \quad (5.21)$$

The integral representations and explicit expressions for the CG coefficients of the tensor product $T_{\tau_1} \otimes T_{\tau_2}$ can be derived from the corresponding results of section 4 by using the substitutions (5.5) and (5.6).

Acknowledgments

We want to thank Professor I H Duru for useful discussions. One of the authors (GAK) would like to thank Professor I H Duru for an invitation to the University of Trakya at Edirne, Turkey. GAK also acknowledges the support of the Scientific and Technical Research Council of Turkey (TÜBİTAK).

Appendix A. Calculation of the sum (3.30)

Let us consider the integral

$$I_c = \frac{1}{2\pi \mathrm{i}} \int_c f(z)\, \mathrm{d}z \tag{A.1}$$

where

$$f(z) = \cot \pi z \frac{\Gamma(a_1 + m_1 + \varepsilon + z)\Gamma(a_2 - m_2 + \varepsilon + z)\Gamma(a + \varepsilon + z)}{\Gamma(1 - a_1 + m_1 + \varepsilon + z)\Gamma(a_2 - m_2 + \varepsilon + z)\Gamma(1 - a + \varepsilon + z)} \tag{A.2}$$

and C is a circle of radius R as large as we please avoiding all zeros of $\sin \pi(a_1+m_1+\varepsilon+z)$ or $\sin \pi(a_2-m_2+\varepsilon+z)$ or $\sin \pi(a+\varepsilon+z)$ or $\sin \pi z$. The series (3.30) is obviously the sum of $f(z)$ at the poles $z = 0, \pm 1, \pm 2, \ldots$, of $\cot(\pi z)$. By arguments exactly parallel of that of [16, ch 1.4] we can show that as $R \to \infty$, $|I_C| \to 0$, provided that $Rl(a_1 + a_2 + a_3) < 1$. However, I_C is equal to the sum of all residues of the integrand at its poles within the contour. Thus, the series S is equal to minus of the residues at the poles of $\Gamma(z+a_1+m_1+\varepsilon)$, $\Gamma(z + a_2 - m_2 + \varepsilon)$ and $\Gamma(z + a + \varepsilon)$. Hence, we find

$$\begin{aligned}
S = \pi \Bigg\{ & \cot \pi(a_1 + m_1 + \varepsilon) \frac{\Gamma(a - a_1 - m_1)\Gamma(a_2 - a_1 - m)}{\Gamma(1 - 2a_1)\Gamma(1 - a - a_1 - m_1)\Gamma(1 - a_2 - a_1 - m)} \\
& \times {}_3F_2 \begin{bmatrix} 2a_1, a + a_1 + m_1, a_2 + a_1 + m; & \\ & 1 \\ 1 - a + a_1 + m_1, 1 - a_2 + a_1 + m; & \end{bmatrix} \\
& + \cot \pi(a_2 - m_2 + \varepsilon) \frac{\Gamma(a - a_2 + m_2)\Gamma(a_1 - a_2 + m)}{\Gamma(1 - 2a_2)\Gamma(1 - a - a_2 + m_2)\Gamma(1 - a_1 - a_2 + m)} \\
& \times {}_3F_2 \begin{bmatrix} 2a_2, a + a_2 - m_2, a_2 + a_1 - m; & \\ & 1 \\ 1 - a + a_2 - m_2, 1 - a_1 + a_2 - m; & \end{bmatrix} \\
& + \cot \pi(a + \varepsilon) \frac{\Gamma(a_1 - a + m_1)\Gamma(a_2 - a - m_2)}{\Gamma(1 - 2a)\Gamma(1 - a_1 - a + m_1)\Gamma(1 - a_2 - a - m_2)} \\
& \times {}_3F_2 \begin{bmatrix} 2a, a + a_1 - m_1, a + a_2 + m_2; & \\ & 1 \\ 1 - a_1 + a - m, 1 - a_2 + a + m_2; & \end{bmatrix} \Bigg\}
\end{aligned} \tag{A.3}$$

where

$${}_3F_2 \begin{bmatrix} a, b, c; & \\ & z \\ d, e; & \end{bmatrix} = \sum_{n=o}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n n!} z^n \tag{A.4}$$

is the generalized hypergeometric function ${}_3F_2(z)$. Here $(a)_n = (\Gamma(a + n)/\Gamma(a))$. The series ${}_3F_2(1)$ is convergent if $Rl(d + e - a - b - c) > 0$.

Appendix B. Relation between ${}_3F_2(1)$ series

In this appendix we introduce Whipple's notation (see section 4.3 of [15]). Let $r_0, r_1, r_2, r_3, r_4, r_5$ be six parameters such that

$$\sum_{i=0}^{5} r_i = 0 \tag{B.1}$$

and let

$$\alpha_{lmn} = 1/2 + r_l + r_m + r_n \qquad \beta_{mn} = 1 + r_m - r_n. \tag{B.2}$$

Note that the coefficients α_{lmn} are totally symmetric whereas $\beta_{mn} = 2 - \beta_{nm}$. The Whipple functions $F_p(l; m, n)$ and $F_n(l; m, n)$ are defined as

$$F_p(l; m, n) = \frac{1}{\Gamma(\alpha_{ghj})\Gamma(\beta_{ml})\Gamma(\beta_{nl})}\, {}_3F_2\left[\begin{matrix}\alpha_{gmn}, & \alpha_{hmn}, & \alpha_{kmn};\\ \beta_{ml} & \beta_{nl}; & 1\end{matrix}\right] \tag{B.3}$$

and

$$F_n(l; m, n) = \frac{1}{\Gamma(\alpha_{lmn})\Gamma(\beta_{lm})\Gamma(\beta_{ln})}\, {}_3F_2\left[\begin{matrix}\alpha_{lhk}, & \alpha_{lgk}, & \alpha_{lgh};\\ \beta_{lm} & \beta_{ln} & 1\end{matrix}\right] \tag{B.4}$$

with the labels (g, h, k, l, m, n) denoting any permutation of $(0, 1, 2, 3, 4, 5)$. The convergence condition for $F_p(l; m, n)$ is $\mathrm{Re}\,(\alpha_{ghj}) > 0$ and the convergence condition for $F_n(l; m, n)$ is $\mathrm{Re}\,(\alpha_{lmn}) > 0$. We note that any $F_n(l; m, n)$ function is obtained from the $F_p(l; m, n)$ function by changing the signs of all the r parameters.

The two-term relation between ${}_3F_2(1)$ functions can be written in the present notation as

$$F_p(l; m, n) = F_p(l; m', n') \tag{B.5}$$

$$F_n(l; m, n) = F_n(l; m', n') \tag{B.6}$$

for any combination of l, m, n, m' and n'. The Whipple functions $F_p(l; m, n)$ and $F_n(l; m, n)$ are thus independent of m and n and will be denoted by $F_p(l)$ and $F_n(l)$, respectively.

All the three-term relations possible between 120 ${}_3F_2(1)$ functions are summed up in the six relations in Whipple's notation

$$\frac{\sin\pi\beta_{23}}{\pi\Gamma(\alpha_{023})} F_p(0) = \frac{F_n(2)}{\Gamma(\alpha_{134})\Gamma(\alpha_{135})\Gamma(\alpha_{345})} - \frac{F_n(3)}{\Gamma(\alpha_{124})\Gamma(\alpha_{125})\Gamma(\alpha_{245})} \tag{B.7}$$

$$\frac{\sin\pi\beta_{45}F_p(0)}{\Gamma(\alpha_{012})\Gamma(\alpha_{013})\Gamma(\alpha_{023})} + \frac{\sin\pi\beta_{50}F_p(4)}{\Gamma(\alpha_{124})\Gamma(\alpha_{134})\Gamma(\alpha_{234})} + \frac{\sin\pi\beta_{54}F_p(5)}{\Gamma(\alpha_{125})\Gamma(\alpha_{135})\Gamma(\alpha_{235})} = 0 \tag{B.8}$$

$$\frac{F_p(0)}{\Gamma(\alpha_{012})\Gamma(\alpha_{013})\Gamma(\alpha_{024})\Gamma(\alpha_{014})\Gamma(\alpha_{034})} + \frac{\sin\pi\beta_{05}F_p(0)}{\Gamma(\alpha_{123})\Gamma(\alpha_{124})\Gamma(\alpha_{134})\Gamma(\alpha_{234})} = R_0F_p(5) \tag{B.9}$$

where $\pi^3R_0 = \sin\pi\alpha_{145}\sin\pi\alpha_{245}\sin\pi\alpha_{345} + \sin\pi\alpha_{123}\sin\pi\beta_{40}\sin\pi\beta_{50}$ and the relations which are obtained by changing the signs of all the r's.

When one of the α_{lmn} parameters is a negative integer the three-term relations reduce to two-term relations between 18 terminating Whipple functions. For the case in which $\alpha_{035} = -k$, k being a positive integer or zero, one has

$$\begin{aligned}\Gamma(\alpha_{014})\Gamma(\alpha_{145})\Gamma(\alpha_{134})F_p(2) &= \Gamma(\alpha_{024})\Gamma(\alpha_{245})\Gamma(\alpha_{234})F_p(1)\\ &= \Gamma(\alpha_{012})\Gamma(\alpha_{125})\Gamma(\alpha_{123})F_p(4)\\ &= (-1)^k\Gamma(\alpha_{014})\Gamma(\alpha_{024})\Gamma(\alpha_{012})F_n(0)\\ &= (-1)^k\Gamma(\alpha_{134})\Gamma(\alpha_{234})\Gamma(\alpha_{123})F_n(3)\\ &= (-1)^k\Gamma(\alpha_{145})\Gamma(\alpha_{245})\Gamma(\alpha_{125})F_n(5).\end{aligned} \tag{B.10}$$

Appendix C. Bilateral series

By the bilateral series we mean the series (see [15, ch 6])

$$ {}_AH_B\left[\begin{matrix} c_1, c_2, \ldots, c_A; \\ \\ d_1, d_2, \ldots, d_B; \end{matrix}\quad z\right] = \sum_{n=-\infty}^{\infty} \frac{(c_1)_n, (c_2)_n, \ldots, (c_A)_n}{(d_1)_n, (d_2)_n, \ldots, (d_B)_n} z^n. \tag{C.1} $$

It has A numerator parameters $c_1, c_2, \ldots, c_A$, B denominator parameters $d_1, d_2, \ldots, d_B$ and one variable z. The function ${}_AH_B(z)$ is defined for all real and complex values of the parameters

$$ c_1, c_2, \ldots, c_A \qquad d_1, d_2, \ldots, d_B \tag{C.2} $$

except zero or integers and for all values of the variable z such that $|z| = 1$. If $z = -1$ we must have

$$ Rl(d_1 + d_2 + \cdots + d_B - c_1 - c_2 - \cdots - c_A) > 1 \tag{C.3} $$

for convergence and if $z = 1$

$$ Rl(d_1 + d_2 + \cdots + d_B - c_1 - c_2 - \cdots - c_A) > 0. \tag{C.4} $$

If any one of the c parameters is a negative integer the series terminates above, if any one of the d parameters is a positive integer the series terminates below. If any one of the c parameters is a positive integer or if any one of the b parameters is a negative integer the series is not defined.

There is an interesting general relation between A series of the type ${}_AH_A(1)$ (see (6.3.2) from [15])

$$ \sum_{\mu=1}^{A} \Gamma\left[\begin{matrix} 1+(b)-b_\mu, & b_\mu-(b)' \\ 1-(d)-b_\mu, & b_\mu-(c) \end{matrix}\right] {}_AH_A\left[\begin{matrix} 1+(c)-b_\mu; \\ \\ 1+(d)-b_\mu; \end{matrix}\quad 1\right] = 0 \tag{C.5} $$

where it is understood that there are A of the b, c and d parameters, and $Rl\sum((d-c)) > 0$. A prime denotes the omission of a zero factor in such a sequence of parameters. For example $b_\mu - (b)'$ indicates the sequence $b_\mu - b_1, b_\mu - b_2, \ldots, b_\mu - b_{\mu-1}, b_\mu - b_{\mu+1}, \ldots, b_A$.

In particular, if $A = 3$, we have

$$ \begin{aligned} &\Gamma\left[\begin{matrix} 1+b_2-b_1, 1+b_3-b_1, b_1-b_2, b_1-b_3 \\ 1-d_1-b_1, 1-d_2-b_1, 1-d_3-b_1, b_1-c_1, , b_1-c_2, b_1-c_3 \end{matrix}\right] \\ &\quad\times {}_3H_3\left[\begin{matrix} 1+c_1-b_1, 1+c_2-b_1, 1+c_3-b_1; \\ \\ 1+d_1-b_1, 1+d_2-b_1, 1+d_3-b_1; \end{matrix}\quad 1\right] \\ &\quad+\Gamma\left[\begin{matrix} 1+b_1-b_2, 1+b_3-b_2, b_2-b_1, b_2-b_3 \\ 1-d_1-b_2, 1-d_2-b_2, 1-d_3-b_2, b_2-c_1, , b_2-c_2, b_2-c_3 \end{matrix}\right] \\ &\quad\times {}_3H_3\left[\begin{matrix} 1+c_1-b_2, 1+c_2-b_2, 1+c_3-b_2; \\ \\ 1+d_1-b_2, 1+d_2-b_2, 1+d_3-b_2; \end{matrix}\quad 1\right] \\ &\quad+\Gamma\left[\begin{matrix} 1+b_1-b_3, 1+b_2-b_3, b_3-b_1, b_3-b_2 \\ 1-d_1-b_3, 1-d_2-b_3, 1-d_3-b_3, b_3-c_1, , b_3-c_2, b_3-c_3 \end{matrix}\right] \\ &\quad\times {}_3H_3\left[\begin{matrix} 1+c_1-b_3, 1+c_2-b_3, 1+c_3-b_3; \\ \\ 1+d_1-b_3, 1+d_2-b_3, 1+d_3-b_3; \end{matrix}\quad 1\right] \end{aligned} \tag{C.6} $$

where

$$\Gamma\begin{bmatrix} c_1, c_2, \ldots, c_A \\ d_1, d_2, \ldots, d_B \end{bmatrix} \equiv \frac{\Gamma(c_1)\Gamma(c_2)\ldots\Gamma(c_A)}{\Gamma(d_1)\Gamma(d_2)\ldots\Gamma(d_B)}. \tag{C.7}$$

This is a relation between three ${}_3H_3(1)$ series.

It follows from (C.5) that, in the general case, the bilateral series ${}_AH_A(1)$ with unit argument can be expressed in terms of the generalized hypergeometric function ${}_AF_{A-1}(1)$ with unit argument. In particular, the function ${}_3H_3(1)$ is expressed in terms of ${}_3F_2(1)$.

Setting $b_1 = d_1$, $b_2 = d_2$ and $b_3 = 1$, we find

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(c_1+n)\Gamma(c_2+n)\Gamma(c_3+n)}{\Gamma(d_1+n)\Gamma(d_2+n)\Gamma(d_3+n)} = \frac{\pi}{\sin\pi c_1 \sin\pi c_2 \sin\pi c_3} \frac{1}{\sin\pi(d_1-d_2)}$$
$$\times\left\{ \frac{\sin\pi d_2}{\Gamma(1-d_1+d_2)\Gamma(1-d_1+d_3)\Gamma(d_1-c_1)\Gamma(d_1-c_3)} {}_3F_2 \right.$$
$$\times\begin{bmatrix} 1+c_1-d_1, 1+c_2-d_1, 1+c_3-d_1; & \\ & 1 \\ 1+d_2-d_1, 1+d_3-d_1; & \end{bmatrix}$$
$$-\frac{\sin\pi d_1}{\Gamma(1+d_1+d_2)\Gamma(1+d_3-d_2)\Gamma(d_2-c_1)\Gamma(d_2-c_2)\Gamma(d_2-c_3)} {}_3F_2$$
$$\left.\times\begin{bmatrix} 1-d_2+c_1, 1-d_2+c_2, 1-d_2+c_3; & \\ & 1 \\ 1+d_1-d_2, 1+d_3-d_2; & \end{bmatrix}\right\}. \tag{C.8}$$

Thus, we have another proof of the summation formula for the series (3.30). In concluding this appendix we give a useful formula for calculation of special values of the CG coefficients

$${}_3H_3\begin{bmatrix} b, c, d; & \\ & 1 \\ 1+a-b, 1+a-c, 1+a-d; & \end{bmatrix}$$
$$= [\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)$$
$$\Gamma(1-\tfrac{1}{2}a)\Gamma(1+\tfrac{1}{2}a)\Gamma(1+\tfrac{3}{2}a-b-c-d)][\Gamma(1+a-c-d)$$
$$\Gamma(1+a-b-d)\Gamma(1+a-b-c)\Gamma(1+\tfrac{1}{2}a-b)\Gamma(1+\tfrac{1}{2}a-c)$$
$$\Gamma(1+\tfrac{1}{2}a-d)\Gamma(1+a)\Gamma(1-a)]^{-1}. \tag{C.9}$$

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